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LETTER TO THE EDITOR

Plane rotor with time-varying magnetic flux: implications for the bound-state Aharonov-Bohm effect

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Abstract. The time-dependent Schrödinger equation is solved exactly for an electron constrained to a circular orbit concentric with a cylinder of smaller radius containing magnetic flux changing in time. The average kinetic angular momentum and energy are changed by the induced electric field in such a way as to satisfy Ehrenfest's theorem. The kinetic angular momentum and energy eigenvalues depend on the instantaneous enclosed flux.

The Aharonov–Bohm (AB) effect (Aharonov and Bohm 1959) has recently been criticised as being a mathematical artifact (Bocchieri and Loinger 1978, 1980), which is obtained by the imposition of improper boundary conditions. The experimental verification of the effect (Chambers 1960, Boersch *et al* 1961, Bayh 1962, Jaklevic *et al* 1965, Matteucci and Pozzi 1978) has putatively been explained in terms of 'leakage flux' and stray magnetic fields (Bocchieri *et al* 1979, 1980, Roy 1980). Although the effect is widely accepted (Wu and Yang 1975, Greenberger 1981, Kobe 1979), there are still differing interpretations of it (Strocchi and Wightman 1974).

A simple model (Merzbacher 1962) which exhibits the AB effect for *bound states* is a plane rotor, an electron confined to a circular orbit, which is concentric with a cylinder of smaller radius containing magnetic flux (Peshkin 1981a). When periodic boundary conditions are imposed on the wavefunction, the energy eigenvalues are changed in a way that depends on the enclosed flux. Flux quantisation can be obtained by demanding that the energy spectrum with flux present is the same as without flux. Byers and Yang (1961) point out that the dependence of the energy levels on the flux when the electron is not in a magnetic field is based on the same principle as discussed by Aharonov and Bohm (1959, 1963). This AB effect for bound states has also been criticised as being unphysical (Bocchieri and Loinger 1981). If the wavefunction is incorrectly allowed to have a discontinuous phase, the AB effect can be made to disappear (Bocchieri and Loinger 1978, 1980) so that the energy eigenvalues are the same as in the absence of flux.

In order to understand the nature of the AB effect better, the plane rotor with time-varying magnetic flux (Peshkin *et al* 1961, Peshkin 1981b, Wilczek 1982, Weisskopf 1961) is considered in this paper. When the flux is initially zero, the wavefunction is periodic and single valued. When the flux in the cylinder changes, the electron experiences an induced electric field by Faraday's law. The induced electric field exerts a torque on the electron and changes its kinetic angular momentum. The induced electric field also does work on the electron and changes its energy. At a given time the energy of the electron depends on the instantaneous flux in the cylinder. The change in the kinetic angular momentum and energy of the electron is a necessary consequence of Ehrenfest's theorem (Yang 1976). When the magnetic flux becomes constant, the static bound-state AB effect is obtained. The time-dependent model therefore explains how the eigenvalues of the energy operator evolve from the state of no flux to the state of constant flux.

An electron is constrained to a circular orbit of radius a which is concentric with an infinitely long cylinder of smaller radius b containing magnetic flux $\Phi(t)$ which in general depends on the time t. The problem is described in cylindrical coordinates (ρ, θ, z). For this geometry the magnetic induction vector $\boldsymbol{B}$ is in the z direction,

$$\boldsymbol{B} = B(t)u(b-\rho)\hat{z} \tag{1}$$

where $u(x)$ is the unit step function which is zero for negative argument and unity for positive argument. When the flux $\Phi(t)$ changes in time, Faraday's law, EMF $= -\dot{\Phi}/c$, gives an induced electric field

$$E_\theta = -\dot{\Phi}/2\pi\rho c \qquad \rho > b. \tag{2}$$

A vector potential $\boldsymbol{A}$ which gives the magnetic induction in equation (1) from $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ is†

$$\boldsymbol{A}(\rho, t) = \Phi(t)(2\pi b^2)^{-1}[(b^2/\rho)u(\rho-b) + \rho u(b-\rho)]\hat{\theta}. \tag{3}$$

By Stokes' theorem the flux is

$$\oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = \Phi \tag{4}$$

where $C = \partial S$ is a curve surrounding the cylinder and $\mathrm{d}\boldsymbol{a}$ is an element of area. The electric field $\boldsymbol{E}$ is

$$\boldsymbol{E} = -\nabla\phi - \partial\boldsymbol{A}/\partial(ct) \tag{5}$$

where ϕ is the scalar potential. For the electric field in equation (2) to be obtained from equations (5) and (3), the scalar potential must satisfy $\phi = 0$.

For the vector potential given in equation (3) and for $\phi = 0$, the time-dependent Schrödinger equation for an electron of mass m and charge q constrained to a circular orbit of radius $a > b$ is

$$(\hbar^2/2ma^2)(-\mathrm{i}\partial/\partial\theta - \alpha(t))^2\psi = \mathrm{i}\hbar\partial\psi/\partial t \tag{6}$$

where the dimensionless arbitrary function of time $\alpha(t)$ is

$$\alpha(t) = q\Phi(t)/2\pi\hbar c. \tag{7}$$

The initial conditions at time $t = 0$ are taken to be zero flux $\alpha(0) = 0$ and a periodic wavefunction $\psi(\theta, 0) = (2\pi)^{-1/2}\exp(in\theta)$, where n is an integer.

To solve this time-dependent equation, the eigenvalue problem for the energy operator (Yang 1976) shall first be solved. The energy operator in this gauge is the same as the Hamiltonian in equation (6), since the scalar potential is zero (Kobe and Smirl 1978). The energy operator eigenvalue problem is

$$(\hbar^2/2ma^2)(-\mathrm{i}\partial/\partial\theta - \alpha(t))^2\psi_n = \varepsilon_n(t)\psi_n \tag{8}$$

† Other vector potentials which are gauge equivalent to equation (3) can be used, but then a non-zero scalar potential is also needed. The gauge invariance of this problem is discussed elsewhere.

where the time t is treated as a parameter. The solution to this eigenvalue problem is the periodic eigenfunction

$$\psi_n(\theta, t) = (2\pi)^{-1/2} \exp(in\theta) \tag{9}$$

which is normalised on $(0, 2\pi)$ and $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. There is no time dependence in ψ_n in equation (9) so it automatically satisfies the initial-state condition. The eigenvalue in equation (8) is

$$\varepsilon_n(t) = (n - \alpha(t))^2 \hbar^2 / 2ma^2 \tag{10}$$

which exhibits the dependence of the energy eigenvalue on the flux (Merzbacher 1962). The flux in equation (8) can have any arbitrary time dependence, and does not have to vary adiabatically. The periodic eigenfunction ψ_n in equation (9) is used, instead of one having a discontinuous phase (Bocchieri and Loinger 1978, 1980, 1981), in order that Ehrenfest's theorem is satisfied.

The solution to the Schrödinger equation in equation (6) is

$$\psi(\theta, t) = \psi_n(\theta, t) \exp\left(-\mathrm{i} \int_0^t \mathrm{d}t' \varepsilon_n(t')/\hbar\right) \tag{11}$$

if the flux is zero at time zero and $\psi(\theta, 0) = \psi_n(\theta, 0)$. At time zero the energy in equation (10) is $\varepsilon_n(0) = n^2\hbar^2/2ma^2$. The wavefunction develops in time from $\psi(\theta, 0)$ to equation (11) and the expectation value of the energy operator is given in equation (10).

If the flux reaches a constant value $\Phi(T)$ for times $t > T$, the energy eigenvalue in equation (10) is a constant. The bound-state AB effect for a static field is seen to emerge naturally from the solution to the time-dependent Schrödinger equation. Dirac's quantisation condition (Dirac 1931, 1948) for electric charge q and magnetic charge g is obtained by setting $\alpha(T)$ equal to an integer and $\Phi = 4\pi g$.

To show more physically that the solution given in equations (9)–(11) is correct, consider Ehrenfest's theorem in a form given by Yang (1976). The kinetic angular momentum operator is (Feynman 1962)

$$\boldsymbol{L} = \boldsymbol{r} \times m\boldsymbol{v} \tag{12}$$

since $m\boldsymbol{v} = (\boldsymbol{p} - q\boldsymbol{A}/c)$ is the kinetic momentum operator (Yang 1976). The kinetic angular momentum in the z direction is $L_z = \rho(p_\theta - qA_\theta/c) = \hbar(-\mathrm{i}\partial/\partial\theta - \alpha(t))$. The average z component of the kinetic angular momentum in the state ψ given by equation (11) is

$$\langle \psi(t) | L_z \psi(t) \rangle = \hbar(n - \alpha(t)). \tag{13}$$

The time rate of change of the average kinetic angular momentum is

$$\mathrm{d}\langle \psi | L_z \psi \rangle / \mathrm{d}t = \langle \psi | \tau_z \psi \rangle \tag{14}$$

by Ehrenfest's theorem (Yang 1976). The z component of the torque operator $\tau_z = (\boldsymbol{r} \times q\boldsymbol{E})_z = q\rho E_\theta$ is (Peshkin *et al* 1961, Peshkin 1981b, Wilczek 1982)

$$\tau_z = -\hbar\dot{\alpha}(t) \tag{15}$$

by equations (13), (7) and (2). The torque operator in equation (15) is independent of the coordinates. Thus, no matter how far away the electron is from the axis, it experiences the same torque. If equation (14) is integrated, the result is equation (13), since the flux at the time zero is zero and the angular momentum at time zero is $\langle\psi(0)|L_z\psi(0)\rangle = n\hbar$†.

The energy operator $\mathscr{E}$ in equation (8) is

$$\mathscr{E} = L_z^2/2I \tag{16}$$

where $\boldsymbol{L}$ is given by equation (12) and the moment of inertia $I = ma^2$. The expectation value of $\mathscr{E}$ in the state ψ in equation (11) is given in equation (10). The time rate of change of the expectation value of $\mathscr{E}$ satisfies Ehrenfest's theorem

$$\mathrm{d}\langle\psi|\mathscr{E}\psi\rangle/\mathrm{d}t = \langle\psi|(L_z/I)\tau_z\psi\rangle \tag{17}$$

by equations (13) and (15). The right-hand side of equation (17) is the power supplied to the electron by the induced electric field and is equal to the average of the quantum power operator (Yang 1976, Kobe *et al* 1982) $P = (q/2)(\boldsymbol{v}\cdot\boldsymbol{E} + \boldsymbol{E}\cdot\boldsymbol{v})$, where $\boldsymbol{v} = (\boldsymbol{p} - q\boldsymbol{A}/c)/m$ is the velocity operator. If equation (17) is integrated from 0 to t, equation (10) is obtained since at time zero the energy is $(n\hbar)^2/2I$. The dependence of the energy eigenvalues on the flux follows from the energy conservation requirement.

In conclusion, I have solved the problem of an electron in a circular orbit which is outside a cylinder with time-varying magnetic flux. As the magnetic field is turned on, an induced electric field from Faraday's law exerts a torque and does work on the electron. The eigenvalues of the z component of the kinetic angular momentum and the energy operators thus depend on the instantaneous flux in such a way that Ehrenfest's theorem is satisfied. The torque is independent of the distance from the axis, so an electron experiences the same torque regardless of where it is. When the magnetic flux becomes static, the energy and kinetic angular momentum eigenvalues depend on the enclosed flux in the same way as for the static AB effect. This result demonstrates that the quantum state of the electron is independent of its past history (Bohm and Hiley 1979).

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References

Aharonov Y and Bohm D 1959 *Phys. Rev.* **115** 485
—— 1963 *Phys. Rev.* **130** 1625
Bayh W 1962 *Z. Phys.* **169** 492
Bocchieri P and Loinger A 1978 *Nuovo Cimento* A **47** 475
—— 1980 *Nuovo Cimento* A **59** 121
—— 1981 *Nuovo Cimento* A **66** 164
Bocchieri P, Loinger A and Siragusa G 1979 *Nuovo Cimento* A **51** 1
—— 1980 *Nuovo Cimento* A **56** 55
Boersch H, Hamisch H, Grohmann K and Wohlleben D 1961 *Z. Phys.* **165** 79
Bohm D and Hiley B J 1979 *Nuovo Cimento* A **52** 295
Byers N and Yang C N 1961 *Phys. Rev. Lett.* **7** 46
Chambers R G 1960 *Phys. Rev. Lett.* **5** 3
Dirac P A M 1931 *Proc. R. Soc.* A **133** 60
—— 1948 *Phys. Rev.* **74** 817
Ehrenberg W and Siday R E 1949 *Proc. Phys. Soc.* B **62** 8

† Weisskopf (1961) incorrectly says that kinetic angular momentum is conserved while the flux is being turned on. In his equation (III.8) he incorrectly assumes that the velocity is constant in time and removes it from the integral. There is an incorrect factor of m in his expression for L on p 66.

Feynman R P 1962 *Quantum Electrodynamics* (New York: Benjamin) p 48
Greenberger D M 1981 *Phys. Rev.* D **23** 1460
Jaklevic R C, Lambe J, Mercereau J E and Silver A H 1965 *Phys. Rev.* **140** A1628
Kobe D H 1979 *Ann. Phys., NY* **123** 381
Kobe D H and Smirl A L 1978 *Am. J. Phys.* **46** 624
Kobe D H, Wen E C-T and Yang K-H 1982 *Phys. Rev.* D to be published
Matteucci G and Pozzi G 1978 *Am. J. Phys.* **46** 619
Merzbacher E 1962 *Am. J. Phys.* **30** 237
Peshkin M 1981a *Phys. Rev.* A **23** 360
—— 1981b *Phys. Rep.* **80** 375
Peshkin M, Talmi I and Tassie L J 1961 *Ann. Phys.,NY* **12** 426
Roy S M 1980 *Phys. Rev. Lett.* **44** 111
Strocchi F and Wightman A S 1974 *J. Math. Phys.* **15** 2198
Weisskopf V F 1961 in *Lectures in Theoretical Physics* ed W E Brittin, B W Downs and J Downs (New York: Interscience) pp 63–70
Wilczek F 1982 *Phys. Rev. Lett.* **48** 1144
Wu T T and Yang C N 1975 *Phys. Rev.* D **12** 3845
Yang K-H 1976 *Ann. Phys., NY* **101** 62